“JUST THE MATHS”

UNIT NUMBER

10.6

DIFFERENTIATION 6
(Inverse trigonometric functions)

by

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UNIT 10.6 - DIFFERENTIATION 6

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

10.6.1 SUMMARY OF RESULTS

The derivatives of inverse trigonometric functions should be considered as standard results. They will be stated here first, before their proofs are discussed.

1.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}},$$
where $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$.

2.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}x] = -\frac{1}{\sqrt{1-x^2}},$$
where $0 \leq \cos^{-1}x \leq \pi$.

3.
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}x] = \frac{1}{1+x^2},$$
where $-\frac{\pi}{2} \leq \tan^{-1}x \leq \frac{\pi}{2}$.

10.6.2 THE DERIVATIVE OF AN INVERSE SINE

We shall consider the formula
$$y = \mathrm{Sin}^{-1}x$$
and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case S in the formula; the reason will be explained later.

The formula is equivalent to
$$x = \sin y,$$
so we may say that
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y \equiv \pm\sqrt{1-\sin^2 y} \equiv \pm\sqrt{1-x^2}.$$

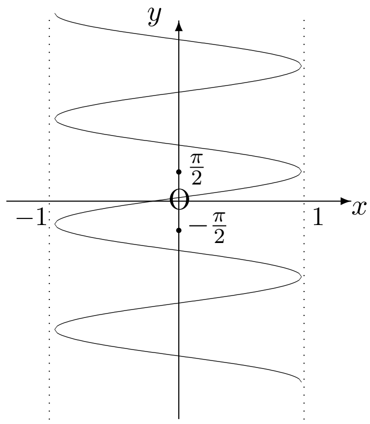
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{\sqrt{1-x^2}}.$$

Consider now the graph of the formula

$$y = \mathrm{Sin}^{-1}x,$$

which may be obtained from the graph of $y = \sin x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i)The variable x must lie in the interval $-1 \leq x \leq 1$;

(ii) For each value of x in the interval $-1 \leq x \leq 1$, the variable y has infinitely many values which are spaced at regular intervals of $\frac{\pi}{2}$.

(iii) For each value of x in the interval $-1 \leq x \leq 1$, there are only two possible values of $\frac{\mathrm{d}y}{\mathrm{d}x}$, one of which is positive and the other negative.

(iv) By restricting the discussion to the part of the graph from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$, there will be only one value of y and one (positive) value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each value of x in the interval $-1 \leq x \leq 1$.

The restricted part of the graph defines what is called the **"principal value"** of the inverse sine function and is denoted by $\sin^{-1}x$ using a lower-case s.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}}.$$

10.6.3 THE DERIVATIVE OF AN INVERSE COSINE

We shall consider the formula

$$y = \text{Cos}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case C in the formula; the reason will be explained later.

The formula is equivalent to

$$x = \cos y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\sin y \equiv \pm\sqrt{1-\cos^2 y} \equiv \pm\sqrt{1-x^2}.$$

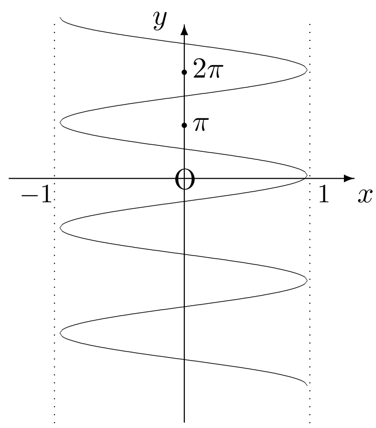
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{\sqrt{1-x^2}}.$$

Consider now the graph of the formula

$$y = \text{Cos}^{-1}x$$

which may be obtained from the graph of $y = \cos x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i) The variable x must lie in the interval $-1 \leq x \leq 1$;

(ii) For each value of x in the interval $-1 \leq x \leq 1$, the variable y has infinitely many values which are spaced at regular intervals of $\frac{\pi}{2}$.

(iii) For each value of x in the interval $-1 \leq x \leq 1$, there are only two possible values of $\frac{\mathrm{d}y}{\mathrm{d}x}$, one of which is positive and the other negative.

(iv) By restricting the discussion to the part of the graph from $y = 0$ to $y = \pi$, we may distinguish the results from those of the inverse sine function; and there will be only one value of y with one (negative) value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each value of x in the interval $-1 \leq x \leq 1$.

The restricted part of the graph defines what is called the **"principal value"** of the inverse cosine function and is denoted by $\cos^{-1}x$ using a lower-case c.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}x] = -\frac{1}{\sqrt{1-x^2}}.$$

10.6.4 THE DERIVATIVE OF AN INVERSE TANGENT

We shall consider the formula

$$y = \mathrm{Tan}^{-1}x$$

and determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note:
There is a special significance in using the upper-case T in the formula; the reason will be explained later.

The formula is equivalent to

$$x = \tan y,$$

so we may say that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \sec^2 y \equiv 1 + \tan^2 y \equiv 1 + x^2.$$

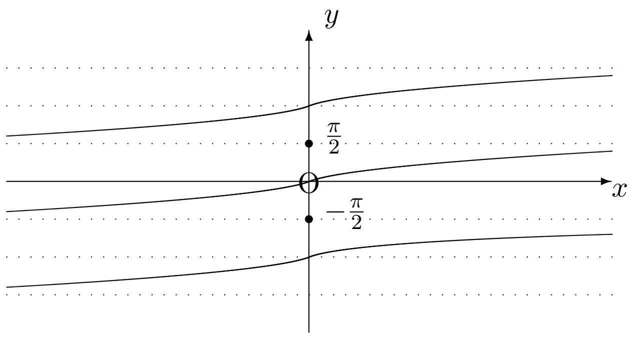
Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}.$$

Consider now the graph of the formula

$$y = \mathrm{Tan}^{-1}x,$$

which may be obtained from the graph of $y = \tan x$ by reversing the roles of x and y and rearranging the new axes into the usual positions. We obtain:



Observations

(i) The variable x may lie anywhere in the interval $-\infty < x < \infty$;

(ii) For each value of x, the variable y has infinitely many values which are spaced at regular intervals of π.

(iii) For each value of x, there is only possible value of $\frac{\mathrm{d}y}{\mathrm{d}x}$, which is positive.

(iv) By restricting the discussion to the part of the graph from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$, there will be only one value of y for each value of x.

The restricted part of the graph defines what is called the **"principal value"** of the inverse tangent function and is denoted by $\tan^{-1}x$ using a lower-case t.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}x] = \frac{1}{1+x^2}.$$

ILLUSTRATIONS

1.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}2x] = \frac{2}{\sqrt{1-4x^2}}.$$

2.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos^{-1}(x+3)] = -\frac{1}{\sqrt{1-(x+3)^2}}.$$

3.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan^{-1}(\sin x)] = \frac{\cos x}{1+\sin^2 x}.$$

4.

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin^{-1}\left(x^5\right)] = \frac{5x^4}{\sqrt{1-x^{10}}} \quad \text{(real only if } \; -1 < x < 1).$$

10.6.5 EXERCISES

1. Determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the following cases, assuming any necessary restrictions on the values of x:

   (a)

$$y = \cos^{-1} 7x;$$

   (b)

$$y = \tan^{-1}(\cos x);$$

   (c)

$$y = \sin^{-1}(3-2x).$$

2. Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\left(\frac{1+\tan x}{1-\tan x}\right)\right] = 1.$$

3. If

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}},$$

   show that

   (a)

$$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = xy + 1;$$

   (b)

$$(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3x\frac{\mathrm{d}y}{\mathrm{d}x} = y.$$

10.6.6 ANSWERS TO EXERCISES

1. (a)

$$-\frac{7}{\sqrt{1-49x^2}} \quad \left(\text{real only if} \quad -\frac{1}{7} < x < \frac{1}{7}\right);$$

(b)

$$-\frac{\sin x}{1+\cos^2 x};$$

(c)

$$-\frac{2}{\sqrt{1-(3-2x)^2}} \quad (\text{real only if} \quad 1 < x < 2).$$