"JUST THE MATHS"

UNIT NUMBER

10.2

DIFFERENTIATION 2
(Rates of change)

by

A.J.Hobson

UNIT 10.2 - DIFFERENTIATION 2

RATES OF CHANGE

10.2.1 INTRODUCTION

The functional relationship

$$y = f(x)$$

can be represented diagramatically by drawing the graph of y against x to obtain, in general, some kind of curve.

Between one point of the curve and another, the values of both x and y will change, in general; and the purpose of this section is to introduce the concept of **the rate of increase of y with respect to x**.

A convenient practical illustration which will provide an aid to understanding is to think of y as the distance travelled by a moving object at time x; because, in this case, the rate of increase of y with respect to x becomes the familiar quantity which we know as **speed**.

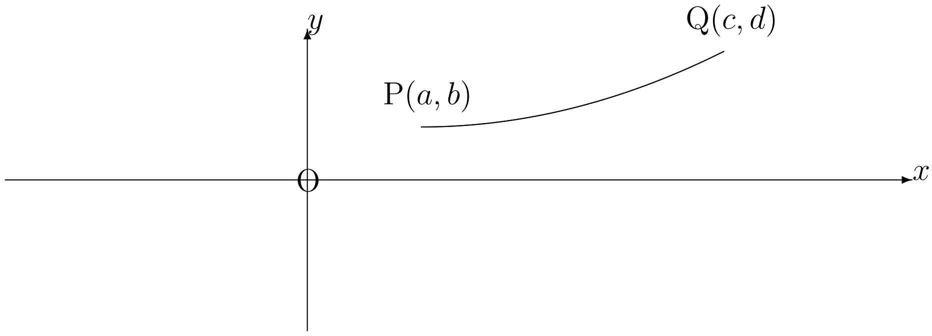
10.2.2 AVERAGE RATES OF CHANGE

Suppose that a vehicle travelled a distance of 280 miles in 7 hours, a journey which is likely to have included short stops, traffic jams, traffic lights and also some fairly high speed motoring. The ratio

$$\frac{280}{7} = 40$$

represents the **"average speed"** of 40 miles per hour over the whole journey. It is a convenient representation of the speed during the journey even though the vehicle might not have been travelling at that speed very often.

Consider now a graph representing the relationship, $y = f(x)$, between two arbitrary variables, x and y, not necessarily time and distance variables.



Between the two points P(a, b) and Q(c, d) an increase of $c - a$ in x gives rise to an increase of $d - b$ in y. Therefore, the average rate of increase of y with respect to x from P to Q is

$$\frac{d - b}{c - a}.$$

If it should happen that y **decreases** as x increases (between P and Q), this quantity will automatically turn out negative; hence,

all rates of increase which are POSITIVE correspond to an INCREASING function,

and

all rates of increase which are NEGATIVE correspond to a DECREASING function.

Note:
For the purposes of later work, the two points P and Q will need to be considered as very close together on the graph, and another way of expressing a rate of increase is to consider notations such as P(x, y) and Q$(x + \delta x, y + \delta y)$ for the pair of points.

Here, we are using the symbols δx and δy to represent **"a small fraction of** x**"** and **"a small fraction of** y**"**, respectively. We **do not** mean δ times x and δ times y. We normally consider that δx is positive, but δy may turn out to be negative.

The average rate of increase in this alternative notation is given by

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}.$$

In other words,

The average rate of increase is equal to

$$\frac{(\text{new value of } y) \quad \text{minus} \quad (\text{old value of } y)}{(\text{new value of } x) \quad \text{minus} \quad (\text{old value of } x)}$$

EXAMPLE
Determine the average rate of increase of the function

$$y = x^2$$

between the following pairs of points on its graph:

(a) $(3, 9)$ and $(3.3, 10.89)$;

(b) $(3, 9)$ and $(3.2, 10.24)$;

(c) $(3, 9)$ and $(3.1, 9.61)$.

Solution

The results are

(a) $\frac{\delta y}{\delta x} = \frac{1.89}{0.3} = 6.3$;

(b) $\frac{\delta y}{\delta x} = \frac{1.24}{0.2} = 6.2$;

(c) $\frac{\delta y}{\delta x} = \frac{0.61}{0.1} = 6.1$

10.2.3 INSTANTANEOUS RATES OF CHANGE

The results of the example at the end of the previous section seem to suggest that, by letting the second point become increasingly close to the first point along the curve, we could determine the **actual** rate of increase of y with respect to x at the first point only, rather than the **average** rate of increase between the two points.

In the above example, the indications are that the rate of increase of $y = x^2$ with respect to x at the point $(3, 9)$ is equal to 6; and this is called the **"instantaneous rate of increase of y with respect to x"** at the chosen point.

The instantaneous rate of increase in this example has been obtained by guesswork on the strength of just three points approaching $(3, 9)$. In general, we need to consider a limiting process in which an **infinite** number of points approach the chosen one along the curve.

This process is represented by

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

and it forms the basis of our main discussion on differential calculus which now follows.

10.2.4 DERIVATIVES

(a) The Definition of a Derivative

In the functional relationship

$$y = f(x)$$

the **"derivative of y with respect to x"** at any point (x, y) on the graph of the function is defined to be the instantaneous rate of increase of y with respect to x at that point.

Assuming that a small increase of δx in x gives rise to a corresponding increase (positive or negative) of δy in y, the derivative will be given by

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

This limiting value is usually denoted by one of the three symbols

$$\frac{\mathrm{d}y}{\mathrm{d}x}, \quad f'(x) \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}[f(x)].$$

Notes:

(i) In the third of these notations, the symbol $\frac{\mathrm{d}}{\mathrm{d}x}$ is called a **"differential operator"**; it cannot exist on its own, but needs to be operating on some function of x. In fact, the first alternative notation is really this differential operator operating on y, which we certainly know to be a function of x.

(ii) The second and third alternative notations are normally used when the derivative of a function of x is being considered without reference to a second variable, y.

(iii) The derivative of a constant function must be zero since the **rate of change** of something which **never changes** is obviously zero.

(iv) Geometrically, the derivative represents the **gradient of the tangent at the point** (x, y) to the curve whose equation is

$$y = f(x).$$

(b) Differentiation from First Principles

Ultimately, the derivatives of **simple** functions may be quoted from a table of standard results; but the establishing of such results requires the use of the definition of a derivative. We illustrate with two examples the process involved:

EXAMPLES

1. Differentiate the function x^4 from first principles.

   Solution

   Here we have a situation where the variable y is not mentioned; so, we could say "let $y = x^4$", and determine $\frac{dy}{dx}$ from first principles in order to answer the question.

   However, we shall choose the alternative notation which does not require the use of y at all.

$$\frac{d}{dx}\left[x^4\right] = \lim_{\delta x \to 0} \frac{(x+\delta x)^4 - x^4}{\delta x}.$$

   Then, from Pascal's Triangle (Unit 2.2),

$$\frac{d}{dx}\left[x^4\right] = \lim_{\delta x \to 0} \frac{x^4 + 4x^3\delta x + 6x^2(\delta x)^2 + 4x(\delta x)^3 + (\delta x)^4 - x^4}{\delta x}$$

$$= \lim_{\delta x \to 0}\left[4x^3 + 6x^2\delta x + 4x(\delta x)^2 + (\delta x)^3\right]$$

$$= 4x^3.$$

   Note:
   This result illustrates a general result which will not be proved here that

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

   for any constant value n, not necessarily an integer.

2. Differentiate the function $\sin x$ from first principles.

   Solution

$$\frac{d}{dx}[\sin x] = \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x},$$

which, from Trigonometric Identities (Unit 3.5), becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}.$$

Finally, using the standard limit (Unit 10.1),

$$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x.$$

Note:
The derivative of $\cos x$ may be obtained in the same way (see EXERCISES 10.2.5, question 2) but it will also be possible to obtain this later (Unit 10.3) by regarding $\cos x$ as $\sin\left(\frac{\pi}{2} - x\right)$.

3. Differentiate from first principles the function

$$\log_b x$$

where b is any base of logarithms.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_b x] = \lim_{\delta x \to 0} \frac{\log_b(x + \delta x) - \log_b x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\log_b\left(1 + \frac{\delta x}{x}\right)}{\delta x}.$$

But writing

$$\frac{\delta x}{x} = r \quad \text{that is} \quad \delta x = rx,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_b x] = \frac{1}{x} \lim_{r \to 0} \frac{\log_b(1 + r)}{r}$$

$$= \frac{1}{x} \lim_{r \to 0} \log_b (1+r)^{\frac{1}{r}}.$$

For convenience, we may choose a base of logarithms which causes the limiting value above to equal 1; and this will occur when

$$b = \lim_{r \to 0} (1+r)^{\frac{1}{r}}.$$

The appropriate value of b turns out to be approximately 2.71828 and this is the standard base of natural logarithms denoted by e.

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x} [\log_e x] = \frac{1}{x}.$$

Note:
In scientific work, the natural logarithm of x is usually denoted by $\ln x$ and this notation will be used in future.

10.2.5 EXERCISES

1. Differentiate from first principles the function $x^3 + 2$.
2. Differentiate from first principles the function $\cos x$.
3. Differentiate from first principles the function $\sqrt{x}$.
   Hint:
   $$(\sqrt{x+\delta x} - \sqrt{x})(\sqrt{x+\delta x} + \sqrt{x}) = \delta x.$$

10.2.6 ANSWERS TO EXERCISES

1. $3x^2$.
2. $-\sin x$.
3. $\frac{1}{2\sqrt{x}}$.