"JUST THE MATHS"

UNIT NUMBER

10.1

DIFFERENTIATION 1
(Functions and limits)

by

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UNIT 10.1 - DIFFERENTIATION 1

10.1.1 FUNCTIONAL NOTATION

Introduction

If a variable quantity, y, depends for its values on another variable quantity, x, we say that **"y is a function of x"** and we write, in general:

$$y = f(x),$$

which is pronounced "y equals f of x".

Notes:

(i) y is called the **"dependent variable"** and x is called the **"independent variable"**. The choice of value for x will be arbitrary, within certain possible constraints, but, once it **is** chosen, the value of y is then fixed.

(ii) The use of the letter f in $f(x)$ is logical because it stands for the word "function"; but once the format of the notation is understood, we can use other letters as appropriate. For example:
the statement $P = P(T)$ could be used to indicate that a pressure, P, is a function of absolute temperature, T;
the statement $i = i(t)$ could be used to indicate that an electric current, i, is a function of time t;
the original statement could have been written $y = y(x)$ without using f at all.

The general format of functional notation may be described as follows:

DEPENDENT VARIABLE =
DEPENDENT VARIABLE(INDEPENDENT VARIABLE)

10.1.2 NUMERICAL EVALUATION OF FUNCTIONS

If α is a number, then $f(\alpha)$ denotes the value of the function $f(x)$ when $x = \alpha$ is substituted into it.

For example, if

$$f(x) \equiv 4\sin 3x,$$

then,

$$f\left(\frac{\pi}{4}\right) = 4\sin\frac{3\pi}{4} = 4 \times \frac{1}{\sqrt{2}} \cong 2.828$$

10.1.3 FUNCTIONS OF A LINEAR FUNCTION

The notation

$$f(ax+b),$$

where a and b are constants, implies a **known** function, $f(x)$, in which x has been replaced by the linear function $ax+b$.

For example, if

$$f(x) \equiv 3x^2 - 7x + 4,$$

then,

$$f(5x-1) \equiv 3(5x-1)^2 - 7(5x-1) + 4;$$

but, in the applications of this kind of notation, it usually best to leave the expression in the bracketed form rather than to expand out the brackets and so lose any obvious connection between $f(x)$ and $f(ax+b)$.

10.1.4 COMPOSITE FUNCTIONS (or Functions of a Function) IN GENERAL

The symbol

$$f[g(x)]$$

implies a **known** function, $f(x)$, in which x has been replaced by **another known** function, $g(x)$.

For example, if

$$f(x) \equiv x^2 + 2x - 5$$

and

$$g(x) \equiv \sin x,$$

then,

$$f[g(x)] \equiv \sin^2 x + 2\sin x - 5;$$

but we can observe also that

$$g[f(x)] \equiv \sin(x^2 + 2x - 5),$$

which is not identical to the first result. Hence, in general,

$$f[g(x)] \not\equiv g[f(x)].$$

There are some exceptions to this, however, as in the case when

$$f(x) \equiv e^x \quad \text{and} \quad g(x) \equiv \log_e x,$$

whereupon we obtain

$$f[g(x)] \equiv e^{\log_e x} \equiv x$$

and

$$g[f(x)] \equiv \log_e (e^x) \equiv x.$$

The functions $\log_e x$ and e^x are said to be **"inverses"** of each other.

10.1.5 INDETERMINATE FORMS

Certain fractional expressions involving functions can become problematic if the values of the variable being substituted into them reduce them to either of the forms

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty}.$$

Both of these forms are meaningless or **"indeterminate"** and need to be dealt with using a concept called **"limiting values"**.

(a) The Indeterminate Form $\frac{0}{0}$

Suppose the fractional expression

$$\frac{f(x)}{g(x)}$$

is such that both $f(x)$ **and** $g(x)$ take the value zero when $x = \alpha$; that is, $f(\alpha) = 0$ and $g(\alpha) = 0$. It is impossible, therefore, to evaluate the fraction when $x = \alpha$; but we may consider its values as x becomes increasingly close to α with out actually reaching it. The standard terminology is to say that **"x tends to α"**, written $x \to \alpha$, for short.

We note that, by the **Factor Theorem**, discussed in Unit 1.8, $(x - \alpha)$ must be a factor of both numerator and denominator; and it turns out that, by cancelling this common factor (which is allowed if x is not going to reach α) we can assign a value to $\frac{f(x)}{g(x)}$ called a limiting value. It still will not be the value of this fraction **at** $x = \alpha$, but represents the value it approaches as x **tends** to α. The result is denoted by

$$\lim_{x \to \alpha} \frac{f(x)}{g(x)}.$$

EXAMPLE
Calculate

$$\lim_{x \to 1} \frac{x - 1}{x^2 + 2x - 3}.$$

Solution

First we factorise the denominator, knowing already that one of its factors must be $x - 1$ because it takes the value zero when $x = 1$.

The result is therefore

$$\lim_{x \to 1} \frac{x-1}{(x-1)(x+3)}$$

$$= \lim_{x \to 1} \frac{1}{x+3}.$$

What we now need to establish is the fixed value which this new fraction approaches as x becomes increasingly close to 1. But since there are no longer any problems with indeterminate forms, we do in fact simply substitute $x = 1$, obtaining the number $\frac{1}{4}$.

Hence,

$$\lim_{x \to 1} \frac{x-1}{x^2+2x-3} = \frac{1}{4}.$$

(b) The Indeterminate Form $\frac{\infty}{\infty}$

This kind of indeterminate form is usually encountered when the value of x itself becomes infinite, either positively or negatively. The object is to evaluate either

$$\lim_{x \to \infty} \frac{f(x)}{g(x)}$$

or

$$\lim_{x \to -\infty} \frac{f(x)}{g(x)}.$$

EXAMPLE
Calculate

$$\lim_{x \to \infty} \frac{2x^2+3x-1}{7x^2-2x+5}.$$

Solution

There is no factorising to do in this type of exercise; we simply divide the numerator and the denominator by the highest power of x appearing, then allow x to become infinite.

The result is therefore

$$\lim_{x\to\infty}\frac{2+\frac{3}{x}-\frac{1}{x^2}}{7-\frac{2}{x}+\frac{5}{x^2}}=\frac{2}{7}.$$

Note:
In the case of the ratio of two polynomials of equal degree, the limiting value as $x\to\pm\infty$ will always be the ratio of the leading coefficients of x. The same principle can be applied to the ratio of two polynomials of unequal degree if we insert zero coefficients in appropriate places to consider them as being of equal degree. The results then obtained will be either zero or infinity.

ILLUSTRATION

$$\lim_{x\to\infty}\frac{5x+11}{3x^2-4x+1}=\lim_{x\to\infty}\frac{0x^2+5x+11}{3x^2-4x+1}=\frac{0}{3}=0.$$

A Useful Standard Limit

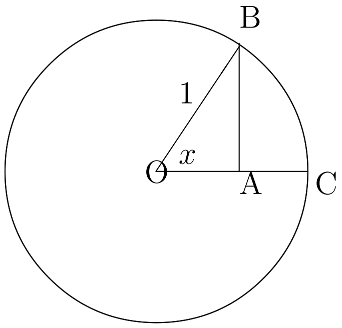
In Unit 3.3, it is shown that, for very small values of x in radians, $\sin x\simeq x$.

This suggests that

$$\lim_{x\to 0}\frac{\sin x}{x}=1,$$

though we may not actually use the result from Unit 3.3 to **prove** the validity of this new limiting value. We shall see later that "$\sin x\simeq x$ for small x" is developed from a calculus technique which **assumes** that $\frac{\sin x}{x}\to 1$ as $x\to 0$; so, we would be using the result to prove itself !

An alternative, non-rigorous, proof is to consider the following diagram in which the angle x is situated at the centre of a circle with radius 1:



In the diagram, the length of line AB $= \sin x$ and the length of arc BC $= x$. Furthermore, as x decreases almost to zero, the two lengths become closer and closer to each other in value. That is,

$$\sin x \to x \quad \text{as} \quad x \to 0$$

or

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

10.1.6 EVEN AND ODD FUNCTIONS

It is easy to see that any **even** power of x will be unchanged in value if x is replaced by $-x$. In a similar way, any **odd** power of x will be unchanged in numerical value, though altered in sign, if x is replaced by $-x$. These two powers of x are examples of an **"even function"** and an **"odd function"** respectively; but the true definition includes a wider range of functions as follows:

DEFINITION
A function $f(x)$ is said to be **"even"** if it satisfies the identity

$$f(-x) \equiv f(x).$$

ILLUSTRATIONS: x^2, $2x^6 - 4x^2 + 5$, $\cos x$.

DEFINITION
A function $f(x)$ is aid to be **"odd"** if it satisfies the identity

$$f(-x) \equiv -f(x).$$

ILLUSTRATIONS x^3, $x^5 - 3x^3 + 2x$, $\sin x$.

Note:
It is not necessary for every function to be either even or odd. For example, the function $x + 3$ is neither even nor odd.

EXAMPLE

Express an arbitrary function, $f(x)$, as the sum of an even function and an odd function.

Solution

We may write

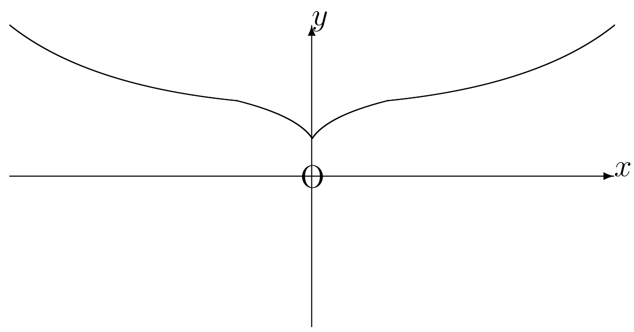
$$f(x) \equiv \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2},$$

in which the first term on the right hand side is unchanged if x is replaced by $-x$ and the second term on the right hand side is reversed in sign if x is replaced by $-x$.

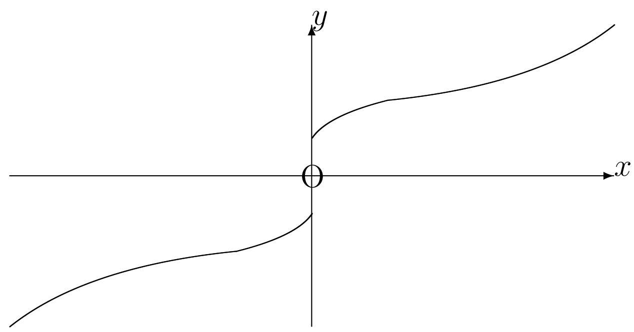
We have thus expressed $f(x)$ as the sum of an even function and an odd function.

(ii) GRAPHS OF EVEN AND ODD FUNCTIONS

(i) The graph of the relationship $y = f(x)$, where $f(x)$ is **even**, will be symmetrical about the y-axis since, for every point (x, y) on the graph, there is also the point $(-x, y)$.



(ii) The graph of the relationship $y = f(x)$, where $f(x)$ is **odd**, will be symmetrical with respect to the origin since, for every point (x, y) on the graph, there is also the point $(-x, -y)$. However, odd functions are more easily recognised by noticing that the part of the graph for $x < 0$ can be obtained from the part for $x > 0$ by reflecting it first in the x-axis and then in the y-axis.

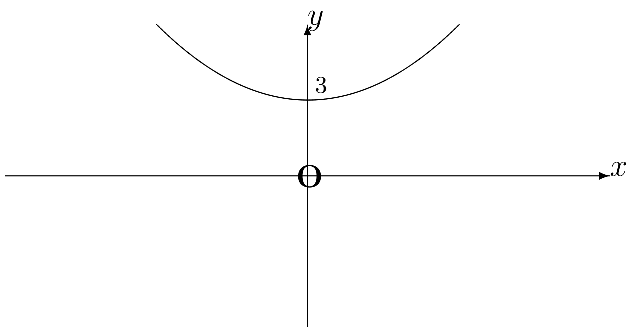


EXAMPLE

Sketch the graph, from $x = -3$ to $x = 3$, of the even function, $f(x)$, defined in the interval $0 < x < 3$ by the formula

$$f(x) \equiv 3 + x^3.$$

Solution



ALGEBRAIC PROPERTIES OF ODD AND EVEN FUNCTIONS

1. The product of an even function and an odd function is an odd function.

Proof:

If $f(x)$ is even and $g(x)$ is odd, then

$$f(-x).g(-x) \equiv f(x).[-g(x)] \equiv -f(x).g(x).$$

2. The product of an even function and an even function is an even function.

Proof:

If $f(x)$ and $g(x)$ are both even functions, then

$$f(-x).g(-x) \equiv f(x).g(x).$$

3. The product of an odd function and an odd function is and even function.

Proof:

If $f(x)$ and $g(x)$ are both odd functions, then

$$f(-x).g(-x) \equiv [-f(x)].[-g(x)] \equiv f(x).g(x).$$

EXAMPLE

Classify the function $f(x) \equiv \sin^4 x.\tan x$ as even, odd or neither even nor odd.

Solution

$$f(-x) \equiv \sin^4(-x).\tan(-x) \equiv \sin^4 x.[-\tan x] \equiv -\sin^4 x.\tan x.$$

The function, $f(x)$, is therefore odd.

10.1.7 EXERCISES

1. If
$$f(x) \equiv 3 + \cos^2\left(\frac{x}{2}\right)$$
determine the values of $f(0), f(\pi), f(\frac{\pi}{2})$.

2. If
$$f(x) \equiv 2x \quad \text{and} \quad g(x) \equiv x^2,$$
verify that
$$f[g(x)] \not\equiv g[f(x)].$$

3. If
$$f(x) \equiv x^2 - 4x + 6,$$
verify that
$$f(2-x) \equiv f(2+x).$$

4. Determine simple functions $f(x)$ and $g(x)$ such that the following functions can be identified with $f[g(x)]$:

   (a)
$$3(x^2+2)^3;$$

   (b)
$$(x^2+1)^{-\frac{1}{2}};$$

   (c)
$$\cos^2 x.$$

5. Evaluate the following limits:

   (a)

$$\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 8x + 7};$$

   (b)

$$\lim_{x \to 2} \frac{x - 2}{x^2 - (x + 2)};$$

   (c)

$$\lim_{x \to \infty} \frac{3x^2 + 5x - 4}{5x^2 - x + 7};$$

   (d)

$$\lim_{r \to -\infty} \frac{(2r + 1)^2}{(r - 1)(r + 3)};$$

   (e)

$$\lim_{\theta \to 0} \frac{\sin 3\theta}{\theta}.$$

6. Express the function e^x as the sum of an even function and an odd function.

7. Sketch the graph, from $x = -5$ to $x = 5$ of the odd function, $f(x)$, defined in the interval $0 < x < 5$ by the formula

$$f(x) \equiv \cos \frac{\pi x}{10}.$$

8. Classify the function

$$f(x) \equiv \tan^2 x + \text{cosec}^3 x . \cos x$$

   as even, odd or neither even nor odd.

10.1.8 ANSWERS TO EXERCISES

1. 4, 3, $\frac{7}{2}$.
2. $2x^2 \not\equiv (2x)^2$.

3. Both are identically equal to $x^2 + 2$.

4. (a)

$$f(x) \equiv 3x^3 \quad \text{and} \quad g(x) \equiv x^2 + 2;$$

(b)

$$f(x) \equiv x^{-\frac{1}{2}} \quad \text{and} \quad g(x) \equiv x^2 + 1;$$

(c)

$$f(x) \equiv x^2 \quad \text{and} \quad g(x) \equiv \cos x.$$

5. (a) $-\frac{2}{3}$;

   (b) $\frac{1}{3}$;

   (c) $\frac{3}{5}$;

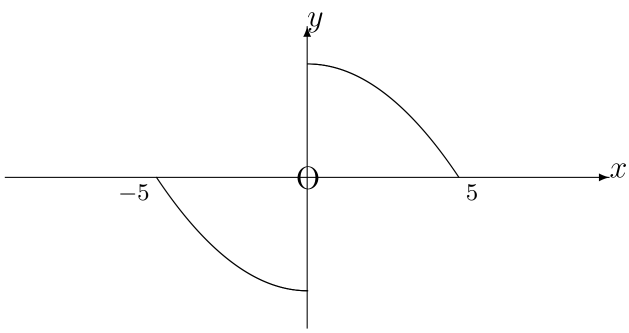
   (d) 4;

   (e) 3.

6.

$$e^x \equiv \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}.$$

7. The graph is as follows:



8. The function is neither even nor odd.