"JUST THE MATHS"

UNIT NUMBER

1.7

ALGEBRA 7
(Simultaneous linear equations)

by

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UNIT 1.7 - ALGEBRA 7 - SIMULTANEOUS LINEAR EQUATIONS

Introduction
When Mathematics is applied to scientific work, it is often necessary to consider several statements involving several variables which are required to have a common solution for those variables. We illustrate here the case of two simultaneous linear equations in two variables x and y and three simultaneous linear equations in three unknowns x, y and z.

1.7.1 TWO SIMULTANEOUS LINEAR EQUATIONS IN TWO UNKNOWNS

Consider the simultaneous linear equations:

$$\begin{aligned} ax + by &= p, \\ cx + dy &= q. \end{aligned}$$

To obtain the solution we first eliminate one of the variables in order to calculate the other. For instance, to eliminate x, we try to make coefficient of x the same in both equations so that, subtracting one statement from the other, x disappears. In this case, we multiply the first equation by c and the second equation by a to give

$$\begin{aligned} cax + cby &= cp, \\ acx + ady &= aq. \end{aligned}$$

Subtracting the second equation from the first, we obtain

$$y(cb - ad) = cp - aq$$

which, in turn, means that

$$y = \frac{cp - aq}{cb - ad} \text{ provided } cb - ad \neq 0.$$

Having found the value of y, we could then either substitute back into one of the original equations to find x or begin again by eliminating y.

However, it is better not to think of the above explanation as providing a **formula** for solving two simultaneous linear equations. Rather, a numerical example should be dealt with from first principles with the numbers provided.

Note:
$cb - ad = 0$ relates to a degenerate case in which the left hand sides of the two equations are proportional to each other. Such cases will not be dealt with at this stage.

EXAMPLE

Solve the simultaneous linear equations

$$\begin{aligned} 6x - 2y &= 1, \qquad (1) \\ 4x + 7y &= 9. \qquad (2) \end{aligned}$$

Solution

Multiplying the first equation by 4 and the second equation by 6,

$$\begin{aligned} 24x - 8y &= 4, \qquad (4) \\ 24x + 42y &= 54. \qquad (5) \end{aligned}$$

Subtracting the second of these from the first, we obtain $-50y = -50$ and hence $y = 1$.

Substituting back into equation (1), $6x - 2 = 1$, giving $6x = 3$, and, hence, $x = \frac{1}{2}$.

Alternative Method

Multiplying the first equation by 7 and the second equation by -2, we obtain

$$\begin{aligned} 42x - 14y &= 7, \qquad (5) \\ -8x - 14y &= -18. \qquad (6) \end{aligned}$$

Subtracting equation (6) from equation (5) gives $50x = 25$ and hence, $x = \frac{1}{2}$.

Substituting into equation (1) gives $3 - 2y = 1$ and hence, $-2y = -2$; that is $y = 1$.

1.7.2 THREE SIMULTANEOUS LINEAR EQUATIONS IN THREE UNKNOWNS

Here, we consider three simultaneous equations of the general form

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3; \end{aligned}$$

but the method will be illustrated by a particular example.

The object of the method is to <u>eliminate</u> one of the variables from two different pairs of the three equations so that we are left with a <u>pair</u> of simultaneous equations from which to calculate the other two variables.

EXAMPLE

Solve, for x, y and z, the simultaneous linear equations

$$\begin{aligned} x - y + 2z &= 9, && (1) \\ 2x + y - z &= 1, && (2) \\ 3x - 2y + z &= 8. && (3) \end{aligned}$$

Solution

Firstly, we may eliminate z from equations (2) and (3) by adding them together. We obtain

$$5x - y = 9. \qquad (4)$$

Secondly, we may eliminate z from equations (1) and (2) by doubling equation (2) and adding it to equation (1). We obtain

$$5x + y = 11. \qquad (5)$$

If we now add equation (4) to equation (5), y will be eliminated to give

$$10x = 20 \quad \text{or} \quad x = 2.$$

Similarly, if we subtract equation (4) from equation (5), x will be eliminated to give

$$2y = 2 \quad \text{or} \quad y = 1.$$

Finally, if we substitute our values of x and y into one of the original equations [say equation (3)] we obtain

$$z = 8 - 3x + 2y = 8 - 6 + 2 = 4.$$

Thus,

$$x = 2, \quad y = 1 \quad \text{and} \quad z = 4.$$

1.7.3 ILL-CONDITIONED EQUATIONS

In the simultaneous linear equations of genuine scientific problems, the coefficients will often be decimal quantitities that have already been subjected to rounding errors; and the solving process will tend to amplify these errors. The result may be that such errors swamp the values of the variables being solved for; and we have what is called an **"ill-conditioned"** set of equations. The opposite of this is a **"well-conditioned"** set of equations and all of those so far discussed have been well-conditioned. But let us consider, now, the following example:

EXAMPLE

The simultaneous linear equations

$$\begin{aligned} x + y &= 1, \\ 1.001x + y &= 2 \end{aligned}$$

have the common solution $x = 1000$, $y = -999$.

However, suppose that the coefficient of x in the second equation is altered to 1.000, which is a mere 0.1%. Then the equations have no solution at all since $x + y$ cannot be equal to 1 and 2 at the same time.

Secondly, suppose that the coefficient of x in the second equation is altered to 0.999 which is still only a 0.2% alteration.

The solutions obtained are now $x = -1000$, $y = 1001$ and so a change of about 200% has occurred in original values of x and y.

1.7.4 EXERCISES

1. Solve, for x and y, the following pairs of simultaneous linear equations:

   (a)
   $$\begin{aligned} x - 2y &= 5, \\ 3x + y &= 1; \end{aligned}$$

   (b)
   $$\begin{aligned} 2x + 3y &= 42, \\ 5x - y &= 20. \end{aligned}$$

2. Solve, for x, y and z, the following sets of simultaneous equations:

   (a)
   $$\begin{aligned} x + y + z &= 0, \\ 2x - y - 3z &= 4, \\ 3x + 3y &= 7; \end{aligned}$$

   (b)
   $$\begin{aligned} x + y - 10 &= 0, \\ y + z - 3 &= 0, \\ x + z + 1 &= 0; \end{aligned}$$

   (c)
   $$\begin{aligned} 2x - y - z &= 6, \\ x + 3y + 2z &= 1, \\ 3x - y - 5z &= 1; \end{aligned}$$

(d)

$$\begin{aligned} 2x - 5y + 2z &= 14, \\ 9x + 3y - 4z &= 13, \\ 7x + 3y - 2z &= 3; \end{aligned}$$

(e)

$$\begin{aligned} 4x - 7y + 6z &= -18, \\ 5x + y - 4z &= -9, \\ 3x - 2y + 3z &= 12. \end{aligned}$$

3. Solve the simultaneous linear equations

$$\begin{aligned} 1.985x - 1.358y &= 2.212, \\ 0.953x - 0.652y &= 1.062, \end{aligned}$$

and compare with the solutions obtained by changing the constant term, 1.062, of the second equation to 1.061.

1.7.5 ANSWERS TO EXERCISES

1. (a) $x = 1$, $y = -2$; (b) $x = 6$, $y = 10$.
2. (a) $x = -\frac{2}{9}$, $y = \frac{23}{9}$, $z = -\frac{7}{3}$;
   (b) $x = 3$, $y = 7$, $z = -4$;
   (c) $x = 3$, $y = -2$, $z = 2$;
   (d) $x = 1$, $y = -4$, $z = -4$;
   (e) $x = 3$, $y = 12$, $z = 9$.
3.

$$x \simeq 0.6087, \quad y \simeq -0.7391$$

compared with

$$x \simeq 30.1304, \quad y \simeq 42.413$$

a change of 4850% in x and 5839% in y.