"JUST THE MATHS"

UNIT NUMBER

1.6

ALGEBRA 6
(Formulae and algebraic equations)

by

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UNIT 1.6 - ALGEBRA 6 - FORMULAE AND ALGEBRAIC EQUATIONS

1.6.1 TRANSPOSITION OF FORMULAE

In dealing with technical formulae, it is often required to single out one of the quantities involved in terms of all the others. We are said to **"transpose the formula"** and make that quantity **"the subject of the equation"**.

In order to do this, steps of the following types may be carried out on both sides of a given formula:

(a) Addition or subtraction of the same value;

(b) Multiplication or division by the same value;

(c) The raising of both sides to equal powers;

(d) Taking logarithms of both sides.

EXAMPLES

1. Make x the subject of the formula

$$y = 3(x + 7).$$

Solution
Dividing both sides by 3 gives $\frac{y}{3} = x + 7$; then subtracting 7 gives $x = \frac{y}{3} - 7$.

2. Make y the subject of the formula

$$a = b + c\sqrt{x^2 - y^2}.$$

Solution
(i) Subtracting b gives $a - b = c\sqrt{x^2 - y^2}$;
(ii) Dividing by c gives $\frac{a-b}{c} = \sqrt{x^2 - y^2}$;
(iii) Squaring both sides gives $\left(\frac{a-b}{c}\right)^2 = x^2 - y^2$;
(iv) Subtracting x^2 gives $\left(\frac{a-b}{c}\right)^2 - x^2 = -y^2$;
(v) Multiplying throughout by -1 gives $x^2 - \left(\frac{a-b}{c}\right)^2 = y^2$;

(vi) Taking square roots of both sides gives

$$y = \pm\sqrt{x^2 - \left(\frac{a-b}{c}\right)^2}.$$

3. Make x the subject of the formula

$$e^{2x-1} = y^3.$$

Solution

Taking natural logarithms of both sides of the formula

$$2x - 1 = 3\ln y.$$

Hence

$$x = \frac{3\ln y + 1}{2}.$$

Note:
A genuine scientific formula will usually involve quantities which can assume only positive values; in which case we can ignore the negative value of a square root.

1.6.2 SOLUTION OF LINEAR EQUATIONS

A Linear Equation in a variable quantity x has the general form

$$ax + b = c.$$

Its solution is obtained by first subtracting b from both sides then dividing both sides by a. That is

$$x = \frac{c-b}{a}.$$

EXAMPLES

1. Solve the equation

$$5x + 11 = 20.$$

Solution

The solution is clearly $x = \frac{20-11}{5} = \frac{9}{5} = 1.8$

2. Solve the equation
$$3 - 7x = 12.$$

   Solution
   This time, the solution is $x = \frac{12-3}{-7} = \frac{9}{-7} \simeq -1.29$

1.6.3 SOLUTION OF QUADRATIC EQUATIONS

The standard form of a quadratic equation is

$$ax^2 + bx + c = 0,$$

where a, b and c are constants and $a \neq 0$.

We shall discuss three methods of solving such an equation related very closely to the previous discussion on quadratic expressions. The first two methods can be illustrated by examples.

(a) By Factorisation
This method depends on the ability to determine the factors of the left hand side of the given quadratic equation. This will usually be by trial and error.

EXAMPLES

1. Solve the quadratic equation
$$6x^2 + x - 2 = 0.$$

   Solution
   In factorised form, the equation can be written

$$(3x + 2)(2x - 1) = 0.$$

   Hence, $x = -\frac{2}{3}$ or $x = \frac{1}{2}$.

2. Solve the quadratic equation
$$15x^2 - 17x - 4 = 0.$$

   Solution
   In factorised form, the equation can be written

$$(5x + 1)(3x - 4) = 0.$$

   Hence, $x = -\frac{1}{5}$ or $x = \frac{4}{3}$

(b) By Completing the square

By looking at some numerical examples of this method, we shall be led naturally to a third method involving a **universal** formula for solving any quadratic equation.

EXAMPLES

1. Solve the quadratic equation

$$x^2 - 4x - 1 = 0.$$

   Solution
   On completing the square, the equation can be written

$$(x - 2)^2 - 5 = 0.$$

   Thus,

$$x - 2 = \pm\sqrt{5}.$$

   That is,

$$x = 2 \pm \sqrt{5}.$$

   Left as it is, this is an answer in **"surd form"** but it could, of course, be expressed in decimals as 4.236 and −0.236.

2. Solve the quadratic equation

$$4x^2 + 4x - 2 = 0.$$

   Solution
   The equation may be written

$$4\left[x^2 + x - \frac{1}{2}\right] = 0$$

   and, on completing the square,

$$4\left[\left(x + \frac{1}{2}\right)^2 - \frac{3}{4}\right] = 0.$$

   Hence,

$$\left(x + \frac{1}{2}\right)^2 = \frac{3}{4},$$

   giving

$$x + \frac{1}{2} = \pm\sqrt{\frac{3}{4}}.$$

That is,

$$x = -\frac{1}{2} \pm \sqrt{\frac{3}{4}}$$

or

$$x = \frac{-1 \pm \sqrt{3}}{2}.$$

(c) By the Quadratic Formula

Starting now with an arbitrary quadratic equation

$$ax^2 + bx + c = 0,$$

we shall use the method of completing the square in order to establish the **general** solution.

The sequence of steps is as follows:

$$a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = 0;$$

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] = 0;$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a};$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}};$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}};$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note:
The quantity $b^2 - 4ac$ is called the **"discriminant"** of the equation and gives either two solutions, one solution or no solutions according as its value is positive, zero or negative.

The single solution case is usually interpreted as a pair of coincident solutions while the no solution case really means no **real** solutions. A more complete discussion of this case arises in the subject of **"complex numbers"** (see Unit 6.1).

EXAMPLES

Use the quadratic formula to solve the following:

1.
$$x^2 + 2x - 35 = 0.$$

Solution

$$x = \frac{-2 \pm \sqrt{4 + 140}}{2} = \frac{-2 \pm 12}{2} = 5 \quad \text{or} \quad -7.$$

2.
$$2x^2 - 3x - 7 = 0.$$

Solution

$$x = \frac{3 \pm \sqrt{9 + 56}}{4} = \frac{3 \pm \sqrt{65}}{4} = \frac{3 \pm 8.062}{4} \simeq 2.766 \quad \text{or} \quad -1.266$$

3.
$$9x^2 - 6x + 1 = 0.$$

Solution

$$x = \frac{6 \pm \sqrt{36 - 36}}{18} = \frac{6}{18} = \frac{1}{3} \quad \text{only}.$$

4.
$$5x^2 + x + 1 = 0.$$

Solution

$$x = \frac{-1 \pm \sqrt{1 - 20}}{10}.$$

Hence, there are no real solutions

1.6.4 EXERCISES

1. Make the given symbol the subject of the following formulae
   (a) x: $a(x-a)=b(x+b)$;
   (b) b: $a=\frac{2-7b}{3+5b}$;
   (c) r: $n=\frac{1}{2L}\sqrt{\frac{r}{p}}$;
   (d) x: $ye^{x^2+1}=5$.
2. Solve, for x, the following equations
   (a) $14x=35$;
   (b) $3x-4.7=2.8$;
   (c) $4(2x-5)=3(2x+8)$.
3. Solve the following quadratic equations by factorisation:
   (a) $x^2+5x-14=0$;
   (b) $8x^2+2x-3=0$.
4. Where possible, solve the following quadratic equations by the formula:
   (a) $2x^2-3x+1=0$; (b) $4x=45-x^2$;
   (c) $16x^2-24x+9=0$; (d) $3x^2+2x+11=0$.

1.6.5 ANSWERS TO EXERCISES

1. (a) $x=\frac{b^2+a^2}{a-b}$;
   (b) $b=\frac{2-3a}{7+5a}$;
   (c) $r=4n^2L^2p$;
   (d) $x=\pm\sqrt{\ln 5-\ln y-1}$.
2. (a) 2.5; (b) 2.5; (c) 22.
3. (a) $x=-7$, $x=2$;
   (b) $x=-\frac{3}{4}$, $x=\frac{1}{2}$.
4. (a) $x=1$, $x=\frac{1}{2}$;
   (b) $x=5$, $x=-9$;
   (c) $x=\frac{3}{4}$ only;
   (d) No solutions.