"JUST THE MATHS"

UNIT NUMBER

1.4

ALGEBRA 4
(Logarithms)

by

A.J.Hobson

UNIT 1.4 - ALGEBRA 4 - LOGARITHMS

1.4.1 COMMON LOGARITHMS

The system of numbers with which we normally count and calculate has a base of 10; this means that each of the successive digits of a particular number correspond to that digit multiplied by a certain power of 10.

For example

$$73,520 = 7 \times 10^4 + 3 \times 10^3 + 5 \times 10^2 + 2 \times 10^1.$$

Note:
Other systems (not discussed here) are sometimes used - such as the binary system which uses successive powers of 2.

The question now arises as to whether a given number can be expressed as a single power of 10, not necessarily an integer power. It will certainly need to be a **positive** number since powers of 10 are not normally negative (or even zero).

It can easily be verified by calculator, for instance that

$$1.99526 \simeq 10^{0.3}$$

and

$$2 \simeq 10^{0.30103}.$$

DEFINITION
In general, when it occurs that

$$x = 10^y,$$

for some positive number x, we say that y is the **"logarithm to base 10"** of x (or " **common logarithm"** of x) and we write

$$y = \log_{10} x.$$

EXAMPLES

1. $\log_{10} 1.99526 = 0.3$ from the illustrations above.
2. $\log_{10} 2 = 0.30103$ from the illustrations above.
3. $\log_{10} 1 = 0$ simply because $10^0 = 1$.

1.4.2 LOGARITHMS IN GENERAL

In practice, with scientific work, only two bases of logarithms are ever used; but it will be useful to include here a general discussion of the definition and properties of logarithms to **any** base so that unnecessary repetition may be avoided. We consider only positive bases of logarithms in the general discussion.

DEFINITION
If B is a fixed positive number and x is another positive number such that

$$x = B^y,$$

we say that y is the **"logarithm to base** B**"** of x and we write

$$y = \log_B x.$$

EXAMPLES

1. $\log_B 1 = 0$ simply because $B^0 = 1$.
2. $\log_B B = 1$ simply because $B^1 = B$.
3. $\log_B 0$ doesn't really exist because no power of B could ever be equal to zero. But, since a very large negative power of B will be a very small positive number, we usually write

$$\log_B 0 = -\infty.$$

1.4.3 USEFUL RESULTS

In preparation for the general properties of logarithms, we note the following two results which can be obtained directly from the definition of a logarithm:

(a) For any positive number x,

$$x = B^{\log_B x}.$$

In other words, any positive number can be expressed as a power of B without necessarily using a calculator.

We have simply replaced the y in the statement $x = B^y$ by $\log_B x$ in the equivalent statement $y = \log_B x$.

(b) For any number y,

$$y = \log_B B^y.$$

In other words, any number can be expressed in the form of a logarithm without necessarily using a calculator.

We have simply replaced x in the statement $y = \log_B x$ by B^y in the equivalent statement $x = B^y$.

1.4.4 PROPERTIES OF LOGARITHMS

The following properties were once necessary for performing numerical calculations before electronic calculators came into use. We do not use logarithms for this purpose nowadays; but we do need their properties for various topics in scientific mathematics.

(a) The Logarithm of Product.

$$\log_B p.q = \log_B p + \log_B q.$$

Proof:
We need to show that, when $p.q$ is expressed as a power of B, that power is the expression on the right hand side of the above formula.

From Result (a) of the previous section,

$$p.q = B^{\log_B p}.B^{\log_B q} = B^{\log_B p + \log_B q},$$

by elementary properties of indices.

The result therefore follows.

(b) The Logarithm of a Quotient

$$\log_B \frac{p}{q} = \log_B p - \log_B q.$$

Proof:
The proof is along similar lines to that in (i).

From Result (a) of the previous section,

$$\frac{p}{q} = \frac{B^{\log_B p}}{B^{\log_B q}} = B^{\log_B p - \log_B q},$$

by elementary properties of indices.

The result therefore follows.

(c) The Logarithm of an Exponential

$$\log_B p^n = n \log_B p,$$

where n need not be an integer.

Proof:
From Result (a) of the previous section,

$$p^n = \left(B^{\log_B p}\right)^n = B^{n \log_B p},$$

by elementary properties of indices.

(d) The Logarithm of a Reciprocal

$$\log_B \frac{1}{q} = -\log_B q.$$

Proof:
This property may be proved in two ways as follows:

Method 1.
The left-hand side $= \log_B 1 - \log_B q = 0 - \log_B q = -\log_B q$.

Method 2.
The left-hand side $= \log_B q^{-1} = -\log_B q$.

(e) Change of Base

$$\log_B x = \frac{\log_A x}{\log_A B}.$$

Proof:
Suppose $y = \log_B x$, then $x = B^y$ and hence

$$\log_A x = \log_A B^y = y \log_A B.$$

Thus,

$$y = \frac{\log_A x}{\log_A B}$$

and the result follows.

Note:
The result shows that the logarithms of any set of numbers to a given base will be directly

proportional to the logarithms of the same set of numbers to another given base. This is simply because the number $\log_A B$ is a constant.

1.4.5 NATURAL LOGARITHMS

It was mentioned earlier that, in scientific work, only two bases of logarithms are ever used. One of these is base 10 and the other is a base which arises **naturally** out of elementary calculus when discussing the simplest available result for the "derivative" (rate of change) of a logarithm.

This other base turns out to be a non-recurring, non-terminating decimal quantity (irrational number) which is equal to 2.71828.....and clearly this would be inconvenient to write into the logarithm notation.

We therefore denote it by e to give the **"natural logarithm"** of a number, x, in the form $\log_e x$, although most scientific books use the alternative notation $\ln x$.

Note:
From the earlier change of base formula we can say that

$$\log_{10} x = \frac{\log_e x}{\log_e 10} \qquad \text{and} \qquad \log_e x = \frac{\log_{10} x}{\log_{10} e}.$$

EXAMPLES

1. Solve for x the indicial equation

$$4^{3x-2} = 26^{x+1}.$$

Solution

The secret of solving an equation where an unknown quantity appears in a power (or index or exponent) is to take logarithms of both sides first.

Here we obtain

$$(3x-2)\log_{10} 4 = (x+1)\log_{10} 26;$$

$$(3x-2)0.6021 = (x+1)1.4150;$$

$$1.8063x - 1.2042 = 1.4150x + 1.4150;$$

$$(1.8603 - 1.4150)x = 1.4150 + 1.2042;$$

$$0.3913x = 2.6192;$$

$$x = \frac{2.6192}{0.3913} \simeq 6.6936$$

2. Rewrite the expression

$$4x + \log_{10}(x+1) - \log_{10} x - \frac{1}{2}\log_{10}(x^3 + 2x^2 - x)$$

as the common logarithm of a single mathematical expression.

Solution

The secret here is to make sure that every term in the given expression is converted, where necessary, to a logarithm with no multiple in front of it or behind it. In this case, we need first to write $4x = \log_{10} 10^{4x}$ and $\frac{1}{2}\log_{10}(x^3 + 2x^2 - x) = \log_{10}\left(x^3 + 2x^2 - x\right)^{\frac{1}{2}}$.

We can then use the results for the logarithms of a product and a quotient to give

$$\log_{10} \frac{10^{4x}(x+1)}{x\sqrt{(x^3 + 2x^2 - x)}}.$$

3. Rewrite without logarithms the equation

$$2x + \ln x = \ln(x - 7).$$

Solution

This time, we need to convert both sides to the natural logarithm of a single mathematical expression in order to remove the logarithms completely.

$$2x + \ln x = \ln e^{2x} + \ln x = \ln xe^{2x}.$$

Hence,

$$xe^{2x} = x - 7.$$

4. Solve for x the equation

$$6\ln 4 + \ln 2 = 3 + \ln x.$$

Solution

In view of the facts that $6\ln 4 = \ln 4^6$ and $3 = \ln e^3$, the equation can be written

$$\ln 2(4^6) = \ln xe^3.$$

Hence,

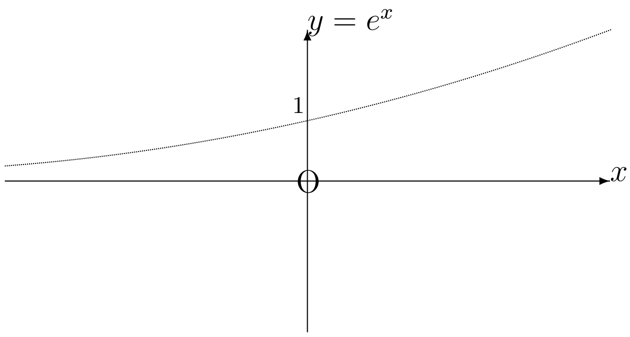
$$2(4^6) = xe^3,$$

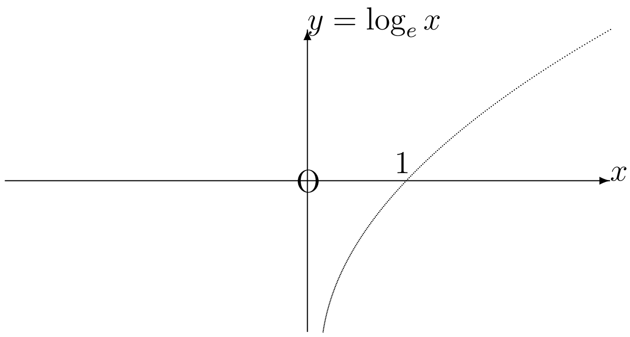
so that

$$x = \frac{2(4^6)}{e^3} \simeq 407.856$$

1.4.6 GRAPHS OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In the applications of mathematics to science and engineering, two commonly used "functions" are $y = e^x$ and $y = \log_e x$. Their graphs are as follows:





They are closely linked with each other by virtue of the two equivalent statements:

$$P = \log_e Q \quad \text{and} \quad Q = e^P$$

for any number P and any <u>postive</u> number Q.

Because of these statements, we would expect similarities in the graphs of the equations

$$y = \log_e x \quad \text{and} \quad y = e^x.$$

1.4.7 LOGARITHMIC SCALES

In a certain kind of graphical work (see Unit 5.3), some use is made of a linear scale along which numbers can be allocated according to their logarithmic distances from a chosen origin of measurement.

Considering firstly that 10 is the base of logarithms, the number 1 is always placed at the zero of measurement (since $\log_{10} 1 = 0$); 10 is placed at the first unit of measurement (since $\log_{10} 10 = 1$), 100 is placed at the second unit of measurement (since $\log_{10} 100 = 2$), and so on.

Negative powers of 10 such as $10^{-1} = 0.1$, $10^{-2} = 0.01$ etc. are placed at the points corresponding to -1 and -2 etc. respectively on an ordinary linear scale.

The logarithmic scale appears therefore in **"cycles"**, each cycle corresponding to a range of numbers between two consecutive powers of 10.

Intermediate numbers are placed at intervals which correspond to their logarithm values.

An extract from a typical logarithmic scale would be as follows:

0.1 0.2 0.3 0.4 1 2 3 4 10

Notes:
(i) A given set of numbers will determine how many cycles are required on the logarithmic scale. For example .3, .6, 5, 9, 23, 42, 166 will require **four** cycles.

(ii) Commercially printed logarithmic scales do not specify the base of logarithms; the change of base formula implies that logarithms to different bases are proportional to each other and hence their logarithmic scales will have the same relative shape.

1.4.8 EXERCISES

1. Without using tables or a calculator, evaluate

   (a) $\log_{10} 27 \div \log_{10} 3$;

   (b) $(\log_{10} 16 - \log_{10} 2) \div \log_{10} 2$.

2. Using properties of logarithms where possible, solve for x the following equations:

   (a) $\log_{10} \frac{7}{2} + 2\log_{10} \frac{4}{3} - \log_{10} \frac{7}{45} = 1 + \log_{10} x$;

   (b) $2\log_{10} 6 - (\log_{10} 4 + \log_{10} 9) = \log_{10} x$.

   (c) $10^x = 5(2^{10})$.

3. From the definition of a logarithm or the change of base formula, evaluate the following:

   (a) $\log_2 7$;

   (b) $\log_3 0.04$;

   (c) $\log_5 3$;

   (d) $3\log_3 2 - \log_3 4 + \log_3 \frac{1}{2}$.

4. Obtain y in terms of x for the following equations:

   (a) $2\ln y = 1 - x^2$;

   (b) $\ln x = 5 - 3\ln y$.

5. Rewrite the following statements without logarithms:

   (a) $\ln x = -\frac{1}{2}\ln(1 - 2v^3) + \ln C$;

   (b) $\ln(1 + y) = \frac{1}{2}x^2 + \ln 4$;

   (c) $\ln(4 + y^2) = 2\ln(x + 1) + \ln A$.

6. (a) If $\frac{I_0}{I} = 10^{ac}$, find c in terms of the other quantities in the formula.

   (b) If $y^p = Cx^q$, find q in terms of the other quantities in the formula.

1.4.9 ANSWERS TO EXERCISES

1. (a) 3; (b) 3.
2. (a) 4; (b) 1; (c) 3.70927
3. (a) 2.807; (b) -2.930; (c) 0.683; (d) 0
4. (a)

$$y = e^{\frac{1}{2}(1-x^2)};$$

(b)

$$y = \sqrt[3]{\frac{e^5}{x}}.$$

5. (a)

$$x = \frac{C}{\sqrt{1-2v^3}};$$

(b)

$$1 + y = 4e^{\frac{x^2}{2}};$$

(c)

$$4 + y^2 = A(x+1)^2.$$

6. (a)

$$c = \frac{1}{a}\log_{10}\frac{I_0}{I};$$

(b)

$$q = \frac{p\log y - \log C}{\log x},$$

using any base.