"JUST THE MATHS"

UNIT NUMBER

1.11

ALGEBRA 11
(Inequalities 2)

by

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UNIT 1.11 - ALGEBRA 11 - INEQUALITIES 2.

1.11.1 RECAP ON MODULUS, ABSOLUTE VALUE OR NUMERICAL VALUE

As seen in Unit 1.1, the Modulus of a numerical quantity ignores any negative signs if there are any. For example, the modulus of -3 is 3, but the modulus of 3 is also 3.

The modulus of an unspecified numerical quantity x is denoted by the symbol

$$\mid x \mid$$

and is defined by the two statements:

$$\mid x \mid = x \quad \text{if} \quad x \geq 0;$$

$$\mid x \mid = -x \quad \text{if} \quad x \leq 0.$$

Notes:
(i) An alternative, but less convenient formula for the modulus of x is

$$\mid x \mid = +\sqrt{x^2}.$$

(ii) It is possible to show that, for any two numbers a and b,

$$\mid a + b \mid \leq \mid a \mid + \mid b \mid .$$

This is called the **"triangle inequality"** and can be linked to the fact that the length of any side of a triangle is never greater than the sum of the lengths of the other two sides.

The proof is a little involved since it is necessary to consider all possible cases of a and b being positive, negative or zero together with a consideration of their relative sizes. It will not be included here.

1.11.2 INTERVAL INEQUALITIES

(a) Using the Modulus notation

In this section, we investigate the meaning of the inequality

$$\mid x - a \mid < k,$$

where a is any number and k is a positive number.

Case 1. $x - a > 0$**.**

The inequality can be rewritten as

$$x - a < k \quad \text{i.e.} \quad x < a + k.$$

Case 2. $x - a < 0$**.**

The inequality can be rewritten as

$$-(x - a) < k \quad \text{i.e.} \quad a - x < k \quad \text{i.e.} \quad x > a - k.$$

Combining the two cases, we conclude that

$$\mid x - a \mid < k \quad \text{means} \quad a - k < x < a + k,$$

which is an open interval having $x = a$ at the centre and extending to a distance of a either side of the centre. A similar interpretation could be given of $\mid x - a \mid \leq k$.

EXAMPLE

Obtain the closed interval represented by the statement

$$\mid x + 3 \mid \leq 10.$$

Solution

Using $a = -3$ and $k = 10$, we have

$$-3 - 10 \leq x \leq -3 + 10.$$

That is,

$$-13 \leq x \leq 7.$$

(b) Using Factorised Polynomials

Suppose a polynomial in x has been factorised into a number of linear factors corresponding to the degree of the polynomial. Then, if certain values of x are substituted in, the polynomial will be positive (or zero) as long as the number of individual factors which become negative is even. Similarly, the polynomial will be negative (or zero) as long as the number of individual factors which become negative is odd.

These observations enable us to find the ranges of values of x for which a factorised polynomial is positive or negative.

EXAMPLE

Find the range of values of x for which the polynomial

$$(x + 3)(x - 1)(x - 2)$$

is strictly positive.

Solution

The first task is to find what are called the **"critical values"**. These are the values of x at which the polynomial becomes equal to zero. In our case, the critical values are $x = -3, 1, 2$.

Next, the critical values divide the x-line into separate intervals where, for the moment, we exclude the critical values themselves. In this case, we obtain

$$x < -3, \quad -3 < x < 1, \quad 1 < x < 2, \quad x > 2.$$

All that is now necessary is to select a value from each of these intervals and investigate how it affects the signs of the factors of the polynomial and hence the sign of the polynomial itself.

$x < -3$ gives (neg)(neg)(neg) and therefore < 0;

$-3 < x < 1$ gives (pos)(neg)(neg) and therefore > 0;

$1 < x < 2$ gives (pos)(pos)(neg) and therefore < 0;

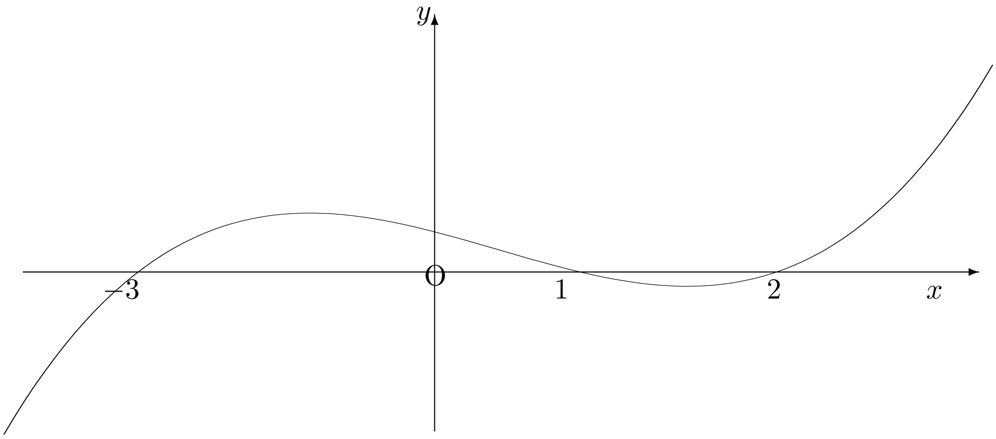
$x > 2$ gives (pos)(pos)(pos) and therefore > 0.

Clearly the critical values will not be included in the answer for this example because they cause the polynomial to have the value zero.

The required ranges are thus

$$-3 < x < 1 \quad \text{and} \quad x > 2.$$

Note:
An alternative method is to sketch the graph of the polynomial as a smooth curve passing through all the critical values on the x-axis. One point only between each critical value will make it clear whether the graph is on the positive side of the x-axis or the negative side. For the above example, the graph is as follows:



1.11.3 EXERCISES

1. Determine the **precise** ranges of values of x which satisfy the following inequalities:
   (a) $| x |< 2$;
   (b) $| x |> 3$;
   (c) $| x - 3 |< 1$;
   (d) $| x + 6 |< 4$;
   (e) $| x + 1 |\geq 2$;
   (f) $0 <| x + 3 |< 5$.
2. Determine the **precise** ranges of values of x which satisfy the following inequalities:
   (a) $(x - 4)(x + 2) \geq 0$;
   (b) $(x + 3)(x - 2)(x - 4) < 0$;
   (c) $(x + 1)^2(x - 3) > 0$;
   (d) $18x - 3x^2 > 0$.
3. By considering the expansion of $(a^2 - b^2)^2$, show that

$$a^4 + b^4 \geq 2a^2b^2.$$

1.11.4 ANSWERS TO EXERCISES

1. (a) $-2 < x < 2$;
   (b) $x < -3$ and $x > 3$;
   (c) $2 < x < 4$;
   (d) $-10 < x < -2$;
   (e) $x \geq 1$ and $x \leq -3$;
   (f) $-8 < x < -3$ and $-3 < x < 2$.
2. (a) $x \leq -2$ and $x \geq 4$;
   (b) $x < -3$ and $2 < x < 4$;
   (c) $x > 3$;
   (d) $0 < x < 6$.